STATS 319: Literature of Statistics
The Sum-of-Squares Algorithmic Paradigm in Statistics
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Lecture 2
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Lecture 2: Mixture Models and SoS Proofs

In this lecture, we will use the sum-of-squares method to develop efficient algorithms for learning mixture models. We'll cover the topics in this order:

1. Introducing the problem of learning Gaussian mixture models,
2. Proof of identifiability for the true sample clusters
3. Obtaining sum-of-squares proofs of identifiability
4. Finding solutions by rounding the pseudoexpectation

Some bibliographic remarks will be deferred to the end.

1 Learning mixture models with separated means

It often make sense to model a distribution $\mathcal{D}$ as a *mixture* of k simpler distributions $\mathcal{D}_1, \dots, \mathcal{D}_k$. That is, we can describe $X \sim \mathcal{D}$ as being sample by first choosing some $i \in [k]$ with probability λ_i, then sampling $X \sim \mathcal{D}_i$. In this lecture, we will be concerned with the problem of estimating the means of each $\mathcal{D}_i$ from samples, as well as the related problem of *clustering* the samples. Formally,

Problem 1.1 (Learning the means of a mixture, and clustering)**.** Let $\mathcal{D}$ be a mixture of k probability distributions $\mathcal{D}_1, \dots, \mathcal{D}_k$ over $\mathbb{R}^d$, with mixing weights $\lambda_1, \dots, \lambda_k$, means (or "centers") $\mu_i = \mathbf{E}_{\mathcal{D}_i} x \in \mathbb{R}^d$ for each $i \in [k]$. Given independent samples $X_1, \cdots, X_n \sim \mathcal{D}$, our goal is to *estimate* μ_i *for each* $i \in [k]$. The related problem of *clustering* asks us to partition $[n]$ into sets $S_1, \dots, S_k$ such that $i \in S_j$ iff $X_i \sim \mathcal{D}_j$.

One may ask under which conditions is the above problem well-defined. Rather than explore this question, we'll restrict our attention to the following special case, for which we will give an algorithm.

Problem 1.2 (Uniform mixture of Δ-separated isotropic Gaussians)**.** This is the special case of Problem 1.1 in which $\lambda_i = \frac{1}{k}$ and $\mathcal{D}_i = \mathcal{N}(\mu_i, \mathbb{1})$ for all $i \in [k]$, and furthermore $\|\mu_i - \mu_j\| \geqslant \Delta$ for all $i \neq j \in [k]$.

Problem 1.2 is known to be information-theoretically possible with $\operatorname{poly}(d, k)$ samples if $\Delta = \Omega(\sqrt{\log k})$ [RV17]. We would like to design an algorithm for this problem which uses only $n = \operatorname{poly}(k, d)$ samples and runs in time $\operatorname{poly}(k, d)$ as well. Our main result in this lecture will be the following:

Theorem 1.3 ([HL18, KSS18])**.** *In the setting of Problem 1.2, there is a universal constant C such that for any even integer t, a degree-t SoS algorithm given $n = (d^t k)^{O(1)}$ samples runs in time $n^{O(1)}$ and with high probability returns $\tilde{\mu}_1, \dots, \tilde{\mu}_k$ such that for all $i \in [k]$,*

$$\|\mu_i - \tilde{\mu}_i\| \leqslant \frac{k 2^{Ct} t^{t/2}}{\Delta^{t-1}}.$$

In particular, if $\Delta \geqslant k^{\gamma}$ for $\gamma > 0$ a fixed constant, then a degree-$O(1/\gamma)$ SoS algorithm can estimate the means up to error $1/\operatorname{poly}(k)$ given polynomially many samples ($d^{O(1/\gamma)} k^{O(1)}$ samples) and in polynomial time ($d^{O(1/\gamma^2)} k^{O(1/\gamma)}$ time). If $\Delta = \Omega(\sqrt{\log k})$, then a degree-$O(\log k)$ SoS algorithm can estimate the means up to error $1/\operatorname{poly}(k)$ given quasi-polynomially many samples in quasi-polynomial time. Prior to this work, the best known polynomial time algorithm required $\Delta = \Omega(k^{1/4})$ [VW02].[1] We'll say more about the history in the bibliographic remarks below.

Remark 1.4. It is noteworthy that their algorithm makes use of higher order ($O(1/\gamma)$) moments of Gaussian (or sub-gaussian) distributions, whereas previous work only used second moments.

The strategy of the algorithm is as follows: first, we establish an SoS proof of identifiability for the clusters $S_1, \dots, S_k$.[2] Then, an SoS relaxation of the appropriate degree is solved, after which we apply a rounding algorithm to recover the true clusters. We'll begin with the proof of identifiability for the clusters.

2 Identifiability for the true clusters

Recall the setup. We have samples $X_1, \dots, X_n$ coming from the uniform mixture over $\mathcal{N}(\mu_1, \mathbb{1}), \dots, \mathcal{N}(\mu_k, \mathbb{1})$, with the clusters $S_1, \dots, S_k$ partitioning $[n]$ so that $S_j = \{i \in [n] \mid X_i \sim \mathcal{N}(\mu_j, \mathbb{1})\}$. In this section we will show that the clusters $S_1, \dots, S_k$ are identifiable from samples with high probability. To begin with, we describe some conditions that are satisfied by the true clusters with high probability.[3]

Cluster conditions. Denote by $\bar{\mu}_j$ the empirical mean of samples in the cluster S_j. For some small $\tau > 0$,

(C1) The size of each cluster is close to its expectation:

$$(1-\tau)\frac{n}{k} \leqslant |S_j| \leqslant (1+\tau)\frac{n}{k}, \ \forall j \in [k]. \tag{1}$$

(C2) The empirical means are close to population means: $\|\bar{\mu}_i - \mu_i\| \leqslant \tau$.

(C3) The empirical moments are subgaussian. To be specific, for large $t \in \mathbb{N}$, we require

$$\frac{1}{|S_j|}\sum_{i \in S_j} \left\langle X_i - \bar{\mu}_j, u \right\rangle^t \leqslant 2t^{t/2}\|u\|^t, \ \forall u \in \mathbb{R}^d, \ j \in [k]. \tag{2}$$

Conditions (C1) and (C2) make sense: we expect that these quantities will concentrate around their means. We comment on the moment condition (C3): note if $\mathcal{D}$ is a sub-gaussian distribution over $\mathbb{R}^d$ with mean vector μ and variance-proxy $\sigma = 1$, then by definition of subgaussianity

$$\mathbb{E}_{X \sim \mathcal{D}} \langle X - \mu, u \rangle^t \leqslant t^{t/2}\|u\|^t, \ \forall u \in \mathbb{R}^d.$$

Hence, this condition basically enforces that the uniform distribution over the samples in S_j is subgaussian, which is what we expect, since the samples in S_j are sampled from $\mathcal{N}(\mu_j, \mathbb{1})$.

Using standard concentration of measure arguments, one can actually show that the above (C1)-(C3) are satisfied with high probability. Surprisingly, it also turns out that these conditions are nearly sufficient for identifying a true cluster, in the sense that, if $S \subset [n]$ satisfies (C1) and (C3), then with high probability there exists a true cluster S_j such that $|S \cap S_j|/|S_j|$ is close to 1. This idea will be made rigorous in Lemma 2.2 below. Before that let us encode the above conditions into the polynomial system $\mathcal{A}$, which will be fully exploited in the SoS proof.

[1] Concurrent with these results is a comparable algorithmic result due to Diakonikolas et al. [DKS18], but it does not use SoS so the theorem statement differs.

[2] The proof of identifiability will use $O(1/\gamma)$-moments.

[3] These are taken from Section 5.2 of [HL18].

System 2.1 (Polynomial constraints on the indicator vector of a cluster)**.** Given samples $X_1, \dots, X_n \in \mathbb{R}^d$ and a small number $\tau > 0$, the following polynomial system $\mathcal{A}$ describes the indicator vector $w \in \{0,1\}^n$ of a cluster S, $w_i = \mathbf{1}_{i \in S}$, as well as the mean of the cluster $\mu \in \mathbb{R}^d$:

1. $w_i^2 = w_i$ for all $i \in [n]$, i.e., w is an indicator vector,
2. $(1-\tau)n/k \leqslant \sum_{i \in [n]} w_i \leqslant (1+\tau)n/k$, enforcing that $|S| \approx n/k$,
3. $\mu \sum_{i \in [n]} w_i = \sum_{i \in [n]} w_i X_i$, meaning that μ is the empirical mean of S,
4. $\sum_{i \in [n]} w_i \langle X_i - \mu, u \rangle^t \leqslant 2t^{t/2} \sum_{i \in [n]} w_i \|u\|^t$, $\forall u \in \mathbb{R}^d$, i.e., the empirical moments are subgaussian.

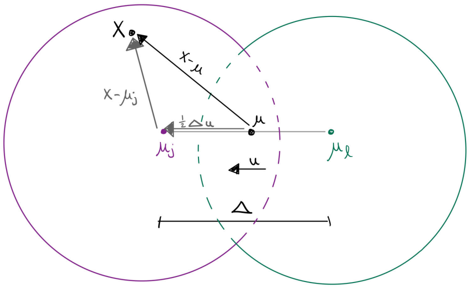
We now show that System 2.1 in conjunction with the conditions (C1)-(C3) ensure that w is an indicator vector for some cluster S_i.

Lemma 2.2 (Lemma 4.20 from [FKP+19])**.** *For $1 \leqslant j \leqslant k$, let a_j be the indicator vector of cluster S_j, and set $A = \sum_{j=1}^k a_j a_j^\top$. Suppose (D1)-(D3) are satisfied by the true clusters. Assume t is a power of 2, and w is a solution of $\mathcal{A}$ with $\tau \leqslant \Delta^{-t}$, then we have*

$$\max_{j \in [k]} \left\langle w, a_j \right\rangle \geqslant \frac{n}{k} \left(1 - \frac{2^{O(t)} t^{t/2} k}{\Delta^t} \right). \tag{3}$$

Notice that since w, a_j are a 0/1 vectors with $\frac{n}{k}(1 \pm \tau)$ nonzero entries, this implies that w and a_j agree on most of their entries when $k 2^{O(t)} \ll (\Delta/\sqrt{t})^t$.

Proof. At a high level, we will use the fact that if w has significant mass on points in both S_j and S_ℓ for $j \neq \ell$, then the uniform distribution over points in the set S indicated by w cannot be subgaussian in the direction $u = \frac{\mu_j - \mu_\ell}{\|\mu_j - \mu_\ell\|}$. To see why, assume for the sake of illustration that S has half of its points in S_j, and half of its points in S_ℓ, in such a way that its mean is equidistant between μ_j and μ_ℓ: $\mu = \frac{1}{2}(\mu_j + \mu_\ell)$.



For points X in S_j,

$$\langle X - \mu, u \rangle = \langle X - \mu_j, \mu \rangle + \langle \tfrac{1}{2}(\mu_j - \mu_\ell), u \rangle \sim N(0,1) + \tfrac{1}{2}\|\mu_i - \mu_j\|,$$

where we've used that subtracting μ is equivalent to subtracting μ_j and adding $\frac{1}{2}(\mu_j - \mu_\ell)$, and that $X - \mu_j \sim \mathcal{N}(0, \mathbb{1})$ so $\langle X - \mu_j, u \rangle \sim \mathcal{N}(0,1)$ for any unit vector u. So the moments of $\langle X - \mu, u \rangle$ will grow at least like $(\frac{1}{2}\Delta)^t$, which will violate the subgaussianity constraint $\mathcal{A}(4)$.

Now, we will implement this intuition in our proof. We will show that

$$\sum_{j \in [k]} \left\langle w, a_j \right\rangle^2 \geqslant \frac{n^2}{k^2} (1 - \varepsilon), \tag{4}$$

For $\varepsilon = 2^{O(t)} t^{t/2} k/\Delta^t$. This is enough to imply our conclusion, because if (4) is true,

$$\max_{j\in[k]} \left\langle w, a_j \right\rangle \geqslant \frac{\sum_{j\in[k]} \left\langle w, a_j\right\rangle^2}{\sum_{j\in[k]}\left\langle w, a_j\right\rangle} \geqslant \frac{1}{\sum_{i=1}^n w_i}\frac{n^2}{k^2}(1-\varepsilon) \geqslant \frac{n}{k}(1-\varepsilon)(1-\tau) \geqslant \frac{n}{k}(1-\varepsilon-\tau), \tag{5}$$

where in the first inequality we use that $\max_{j\in[k]}\langle w, a_\ell\rangle \cdot \sum_{j\in[k]}\langle w, a_\ell\rangle \geqslant \sum_{j\in[k]}\langle w, a_j\rangle^2$, in the second inequality we applied (4), and finally we used the constraint that $\sum w_i = \frac{n}{k}(1 \pm \tau)$.

Now, we'll prove (4). First, by applying constraint (2) in $\mathcal{A}$,

$$\left(\sum_{i\in[k]} \langle w, a_i \rangle\right)^2 = \left(\sum_{i=1}^n w_i\right)^2 \geqslant (1-\tau)^2\,\frac{n^2}{k^2} \geqslant (1-2\tau)\,\frac{n^2}{k^2}.$$

The left-hand side include the left-hand side of (4) as well as cross-terms $\langle w, a_j\rangle\langle w, a_i\rangle$, so it will suffice to show that these cross-terms do not contribute more than $O(\varepsilon)$ to the total,

$$\sum_{j\neq\ell} \langle w, a_j\rangle\langle w, a_\ell\rangle \leqslant \frac{n^2}{k^2} O(\varepsilon). \tag{6}$$

This is where our intuition about the subgaussianity in the direction $u = (\mu_j - \mu_\ell)/\|\mu_j - \mu_\ell\|$ comes in. Using that $\|\mu_j - \mu_\ell\|/\Delta \geqslant 1$,

$$\begin{align}
\langle w, a_j\rangle\langle w, a_\ell\rangle &\leqslant \frac{\|\mu_j - \mu_\ell\|^t}{\Delta^t}\langle w, a_j\rangle\langle w, a_\ell\rangle \tag{7}\\
&= \frac{1}{\Delta^t}\cdot\langle w, a_j\rangle\langle w, a_\ell\rangle\langle \mu_j - \mu_\ell, u\rangle^t \notag\\
&= \frac{1}{\Delta^t}\cdot\langle w, a_j\rangle\langle w, a_\ell\rangle\left(\left\langle \mu_j - \mu, u\right\rangle + \langle \mu - \mu_\ell, u\rangle\right)^t \notag\\
&\overset{(i)}{\leqslant} \frac{2^{t-1}}{\Delta^t}\langle w, a_j\rangle\langle w, a_\ell\rangle\left(\left\langle \mu_j - \mu, u\right\rangle^t + \langle \mu - \mu_\ell, u\rangle^t\right) \notag\\
&= \frac{2^{t-1}}{\Delta^t}\langle w, a_\ell\rangle\sum_{i\in S_j} w_i\cdot\langle \mu_j - \mu, u\rangle^t + \frac{2^{t-1}}{\Delta^t}\langle w, a_j\rangle\sum_{i\in S_\ell} w_i\cdot\langle \mu_\ell - \mu, u\rangle^t, \tag{8}
\end{align}$$

where in (i) we used the triangle inequality: $(a+b)^t \leqslant 2^{t-1}(a^t + b^t)$. Now, we bound just one of the terms above (as they are symmetric). We will introduce the samples, twice use subgaussianity:

$$\begin{align}
\sum_{i\in S_j} w_i\cdot\langle \mu_j - \mu, u\rangle^t &= \sum_{i\in S_j} w_i\cdot\left(\langle \mu_j - X_i, u\rangle + \langle X_i - \mu, u\rangle\right)^t \notag\\
&\leqslant 2^{t-1}\sum_{i\in S_j}\langle \mu_j - X_i, u\rangle^t + 2^{t-1}\sum_{i\in S_j} w_i\cdot\langle X_i - \mu, u\rangle^t. \tag{9}
\end{align}$$

Since by (C2) we have $\|\bar{\mu}_j - \mu_j\| \leqslant \tau$ and by (C3) we have empirical subgaussianity within S_j, using the triangle inequality, the first sum we may bound by $|S_j|\cdot(2^t t^{t/2} + (2\tau)^t) \leqslant (1+\tau)\frac{n}{k}(2^t t^{t/2} + (2\tau)^t)$. For the second sum, we use the polynomial subgaussianity constraint $\mathcal{A}(4)$ to conclude that

$$\sum_{i\in S_j} w_i\cdot\langle X_i - \mu, u\rangle^t \leqslant \sum_{r\in[k]}\sum_{i\in S_r} w_i\cdot\langle X_i - \mu, u\rangle^t \leqslant 2t^{t/2}\sum_{i\in[n]} w_i \leqslant 2t^{t/2}\frac{n}{k}(1+\tau).$$

Together, this gives us that the right-hand side of (9) is at most $(1+\tau)(2^{t+1}t^{t/2} + (2\tau)^t)\frac{n}{k}$, and in turn the right-hand side of (8) is at most $2^{2t+2}\Delta^{-t}(\langle w, a_j\rangle + \langle w, a_\ell\rangle)\frac{n}{k}t^{t/2}$ (we have used that τ is small).

So putting it all together,

$$\sum_{j\neq\ell}\langle w,a_j\rangle\langle w,a_\ell\rangle \leqslant \frac{2^{2t+2}}{\Delta^t}\frac{n}{k}t^{t/2}\sum_{j\neq\ell}\langle w,a_\ell\rangle+\langle w,a_j\rangle \leqslant k\cdot\frac{2^{2t+2}}{\Delta^t}\frac{n}{k}t^{t/2}\cdot 2\sum_{i\in[n]}w_i \leqslant \frac{2^{2t+4}}{\Delta^t}\frac{n^2}{k}t^{t/2} = O(\varepsilon)\frac{n^2}{k^2},$$

as desired. □

3 SoS-izing the proof of identifiability

Lemma 2.2 immediately suggests a procedure for recovering the ground-truth partition $S_1,\dots,S_k$: First find a solution (w,S) to $\mathcal{A}$. According to Lemma 2.2, S should be very close to some S_j. Then we remove the points in S and repeat the above procedure for the remaining points, until we obtain k clusters. However, this does not result an efficient algorithm. Following the usual sum-of-squares paradigm, we will instead solve for a pseudoexpectation $\tilde{\mathbb{E}}$ of sufficiently high degree that satisfies $\mathcal{A}$, and round this pseudoexpectation to find a good clustering.

Encoding the subgaussian condition. Note that we cannot directly make use of $\mathcal{A}$ to find a degree-$O(t)$ pseudoexpectation operator in polynomial time, since there are infinitely many inequality constraints in $\mathcal{A}(4)$ (one for each $u\in\mathbb{R}^d$). Although in the proof above we only used the t-th empirical moments are bounded in the $\binom{k}{2}$ directions of $\mu_j-\mu_\ell, j\neq\ell\in[k]$, this is still problematic because the μ_j's are unknown parameters that we are trying to estimate, so we don't have access to them when we are trying to encode a polynomial system as part of our algorithm. To deal with this issue, we introduce the notion of a "t-explicitly bounded distribution" (also known as *t-certifiably subgaussian* in the literature) below:

Definition 3.1 (t-explicitly bounded)**.** Let $\mathcal{D}$ be a distribution over $\mathbb{R}^d$ with mean μ. For $\sigma>0$ and $t\in\mathbb{N}$, we say that $\mathcal{D}$ is *t-explicitly bounded with variance proxy σ* if for every even number $s\leqslant t$, there is a degree-s SoS proof of the inequality:

$$\vdash_s \mathbb{E}_{X\sim\mathcal{D}}\langle X-\mu,u\rangle^s \leqslant (\sigma s)^{s/2}\|u\|^s. \tag{10}$$

Equivalently, the polynomial $(\sigma s)^{s/2}\|u\|^s-\mathbb{E}_{X\sim\mathcal{D}}\langle X-\mu,u\rangle^s$ can be written as a sum of squares. In this lecture we will assume $\sigma=1$ and just call the distribution t-explicitly bounded, we also assume that t is even to avoid some technical difficulties.

Remark 3.2 (Examples of t-explicitly bounded distributions)**.** Any normal distribution with identity covariance matrix is t-explicitly bounded for any $t\in\mathbb{N}$. The rotation of product distributions with bounded t-th moments are also t-explicitly bounded.

Moreover, [KSS18] proved that σ-Poincaré distributions are t-explicitly bounded. We say a distribution $\mathcal{D}$ is σ-Poincaré if it satisfies the following Poincaré inequality: For all differentiable functions $f:\mathbb{R}^d\longrightarrow\mathbb{R}$,

$$\mathbf{Var}_{X\sim\mathcal{D}}\left[f(X)\right] \leqslant \sigma^2\mathbb{E}_{X\sim\mathcal{D}}\left[\|\nabla f(X)\|^2\right]. \tag{11}$$

Together, these examples comprise many commonly considered distributions.

We'll briefly describe the proof that for any $\zeta\in\mathbb{R}^d$, the distribution $\mathcal{N}(\zeta,\mathbb{1})$ is t-explicitly bounded. This boils down to the fact that the following matrix inequality holds for any $a\leqslant t/2$:

$$\mathbf{E}_{X\sim\mathcal{N}(\zeta,\mathbb{1})}(X-\zeta)^{\otimes a}((X-\zeta)^{\otimes a})^\top \preceq \mathbf{E}_{X\sim\mathcal{N}(0,\mathbb{1})}X^{\otimes a}(X^{\otimes a})^\top. \tag{12}$$

The fact that this implies t-subgaussian behavior is given by taking the quadratic form of the left- and right-hand side with $u^{\otimes a}$ for any unit vector $u \in \mathbb{R}^d$.

So in order to encode the constraint that $\sum_i w_i(X_i - \mu)$ is t-subgaussian, we will replace $\mathcal{A}(4)$ with the polynomial constraint

$$\sum_{i\in[n]} w_i((X_i - \mu)^{\otimes t/2})((X_i - \mu)^{\otimes t/2}) = 2 \cdot \left(\sum_{i\in[n]} w_i\right) \cdot \mathbf{E}_{X\sim\mathcal{N}(0,\mathbb{1})} X^{\otimes t/2}(X^{\otimes t/2})^\top - BB^\top,$$

for B a matrix of indeterminate variables of dimension $d^{t/2}\times d^{t/2}$. This constraint encodes the t-subgaussianity of the w-cluster S as a set of $d^{O(t)}$ polynomial equalities, and the fact that it is feasible follows from the fact that each $\mathcal{D}_j$ is Gaussian, plus an argument that (12) is satisfied (up to a factor of 2) by the emprical samples X_i for $i \in S_j$ with high probability so long as $n = d^{\Omega(t)}$; this is ensured by the condition $n = d^{\Omega(t)}$ in Theorem 1.3.[4] Call this new system of equations $\widehat{\mathcal{A}}$. Since $\widehat{\mathcal{A}}$ has only $d^{O(t)} + \mathrm{poly}(n)$ constraints, finding a pseudoexpectation $\tilde{\mathbb{E}}$ which satisfies $\widehat{\mathcal{A}}$ takes $d^{O(t)} + \mathrm{poly}(n)$ time.

After introducing the new polynomial system $\widehat{\mathcal{A}}$, one can prove a SoS version of Lemma 2.2.

Lemma 3.3 (Lemma 5.3 from [HL18])**.** *Under the same assumptions as Lemma 2.2, let $\tilde{\mathbb{E}}$ be a degree-$O(t)$ pseudoexpectation that satisfies $\widehat{\mathcal{A}}$, then*

$$\tilde{\mathbb{E}} \sum_{j\in[k]} \left\langle w, a_j \right\rangle^2 \geqslant \frac{n^2}{k^2}\left(1 - \frac{2^{O(t)}t^{t/2}k}{\Delta^t}\right). \tag{13}$$

Sketch of proof. Notice that each inequality that appeared in the proof of (4) in Lemma 2.2 can be SoS-ized by applying the usual SoS tools, including SoS versions of Cauchy-Schwarz, Hölder's inequality, and the triangle inequality. □

4 Rounding the pseudomoments

Finally, we will show that we can use our pseudoexpectation satisfying $\widehat{\mathcal{A}}$ to recover the cluster centers. Here, we will make one final modification to our algorithm: we will search for the pseudoexpectation satisfying $\widehat{\mathcal{A}}$ which minimizes the Frobenius norm $\|\tilde{\mathbf{E}}ww^\top\|_F$. This is a convex objective, so we can solve for $\tilde{\mathbf{E}}$ in polynomial time.

Lemma 4.1. *For the degree-$O(t)$ pseudoexpectation operator $\tilde{\mathbf{E}}$ satisfying $\widehat{\mathcal{A}}$ which minimizes $\|\tilde{\mathbf{E}}ww^\top\|_F$, the matrix $M = \tilde{\mathbf{E}}ww^\top$ is close to the block matrix $A = \frac{1}{k}\sum_{j\in[k]} a_i a_i^\top$, in the sense that $\|A - M\|_F^2 \leqslant \varepsilon\|A\|_F^2$ for $\varepsilon = \frac{2^{O(t)}kt^{t/2}}{\Delta^t}$.*

Once we prove this claim, the algorithm is easy: A is a block matrix whose k blocks correspond exactly to the k clusters, and A and M agree on all but an ε-fraction of entries. So $M = \tilde{\mathbf{E}}ww^\top$ is essentially a block matrix whose blocks correspond to the clusters, and we can effectively read these off. Finally, to estimate the mean μ_j, one can take the empirical mean of the samples in S_j.

Proof of Lemma 4.1. Note that $M \succeq 0$, and $\operatorname{Tr} M = \tilde{\mathbf{E}}\sum_{i\in[n]} w_i^2 = \frac{n}{k}(1 \pm \tau)$ by $\mathcal{A}(1)$ and (2).

Now, we have that

$$\|M - A\|_F^2 = \|M\|_F^2 + \|A\|_F^2 - 2\langle M, A\rangle$$

[4] See Lemma 4.1 in [HL18].

$$\leqslant 2(\|A\|_F^2 - \langle M, A\rangle),$$

where the inequality follows because $\tilde{\mathbb{E}}$ was chosen to minimize the Frobenius norm of M, and A corresponds to the (with high probability) feasible choice of $\tilde{\mathbb{E}}$ as the actual expectation of the distribution where w is chosen from the uniform mixture over $\{a_j\}_{j\in[k]}$. But now, notice that

$$\langle M, A\rangle = \frac{1}{k}\tilde{\mathbb{E}}\left[\sum_{j\in[k]}\langle w, a_j\rangle^2\right] \geqslant \frac{n^2}{k^3}(1-\varepsilon),$$

for $\varepsilon = 2^{O(t)}t^{t/2}k/\Delta^t$, where to obtain the inequality we have applied Lemma 3.3, the SoS-version of Lemma 2.2. The lemma now follows because $\|A\|_F^2 = (1 \pm \tau)\frac{n^2}{k^3}$. □

5 Conclusion

Putting it all together, we have now seen the proof of Theorem 1.3. We note that the algorithms presented here can be generalized to the case of non-uniform mixing weights, and to the case when the mixture is over any $\mathcal{D}_1, \dots, \mathcal{D}_k$ which are t-explicitly bounded. By results of [KSS18], the property of being t-explicitly bounded holds for the large family of distributions which satisfy a Poincaré inequality.

Bibliographic remarks. The algorithm given here is based on the concurrent works of Hopkins-Li and Kothari-Steinhardt [HL18, KSS18]; here we are borrowing from the presentations of [HL18, FKP+19].

The study of Problem 1.1 can be traced back to Pearson [Pea94]. Prior to these works, the best algorithm for learning Gaussian mixture model with isotropic components required $\Delta \geqslant k^{1/4}$, via *single-linkage clustering*, which is a simple greedy algorithm. In this parameter regime, every pair of samples from the same cluster are closer to each other in Euclidean distance than are every pair of samples from distinct clusters (with high probability), so the clusters can be identified using this information about sample second moments. The single-linkage clustering algorithm of Vempala and Wang [VW02] was built upon several pioneering works [Das99, DS07, AK+05].

Standard information-theoretic arguments [RV17] show that it's possible to identify the cluster means from $n = \mathrm{poly}(k, d)$ samples when Δ is $\Omega(\sqrt{\log k})$, but prior to 2018 only exponential-time algorithms were known. As is evident from this timeline, this computational-to-statistical gap stood open for a long time until the breakthrough works of Hopkins and Li [HL18], Kothari, Steinhardt and Steurer [KSS18], and Diakonikolas, Kane, and Stewart [DKS18]. The two former results use SoS algorithms, while the latter uses a spectral filtering approach and is significantly different from the approach here.

The SoS and pseudoexpectation inequalities needed in the proof of Lemma 3.3 may be found in Section 7 of [HL18] and Appendix A of [BKS14].

Contact. Comments are welcome at tselil@stanford.edu.

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